Indian J Phys. **82** (9), 1139-1158 (2008)

Two-body problem with retarded interactions and radiation reaction in classical electrodynamics

Yurij Yaremko
Institute for Condensed Matter Physics
1 Svientsitskii St , 79011 Lviv Ukraine

E mail yar@ph icmp lviv ua

Received 26 June 2008 accepted 14 July 2008

Abstract Energy-momentum and angular momentum carried by electromagnetic field of two point-like charged particles are presented Apart from usual contributions to the Noether quantities produced separately by particle 1 and particle 2, the conservation laws contain also the joint contribution due to the fields of both particles So, radiative part of the energy-momentum contains apart from usual integrals of Larmor terms, also the work done by Lorentz forces of point-like charges acting on one another Interference part of radiated angular momentum is the sum of integrals of torque Lorentz forces over particles' world lines Analysis of energy-momentum and angular momentum balance equations results the Lorentz-Dirac equation as an equation of motion for a pointed charge under the influence its own electromagnetic field as well as field produced by the other charge The radiative component of mixed part of Maxwell energy-momentum density is given

Keywords Two-body problem Radiation Reaction Retarded interaction, Lorentz-Dirac equation

PACS Nos. 03 50 De, 11 10 Gh, 11 30 GP

1. Introduction

The most natural and widely accepted equation of motion for a charge when radiation reaction is taken into account is the Lorentz-Dirac equation [1] This equation has been discussed mainly for the case of one charge in an external electromagnetic field In the present paper we consider an isolated system of two point electric charges and their electromagnetic field We study the electromagnetic energy-momentum and angular momentum radiated by charges, the study of energy-momentum and angular momentum balance equations implies the Lorentz-Dirac equation for more than one charge

The dynamics of the entire system is governed by the action

$$I=\sum_{a=1}^{2}\left(-m_a\int ds_a\sqrt{-\dot z_a^2}+e_a\int ds_a A_{a\,\mu}\dot z_a^{\mu}\right)-\frac{1}{16\pi}\int d^4y\, f_{\mu\nu}f^{\mu\nu} \tag{1.1}$$

where $f_{\mu\nu}=\sum_a\left(\partial_\mu A_{a\,\nu}-\partial_\nu A_{a\,\mu}\right)$ is the total field generated by two charges. Charge e_a moves on a world line $\zeta_a\in\mathcal{M}_4$ described by functions $z_a^{\mu}(s_a)$ which give the particle's coordinates as functions of proper time s_a, $\dot z_a^{\mu}=dz_a^{\mu}/ds_a$ is the a-th 4-velocity.

Variation on field variables A_a^{α} yields the Maxwell equations. The retarded and advanced Green's functions are defined globally in the entire spacetime, their convolution with δ-like charge distributions yield the well-known Liénard-Wiechert potentials. The action (1.1) is invariant under ten infinitesimal transformations which constitute Poincaré group. According to Noether theorem, these symmetry properties imply conservation laws, *i.e.* those qunatities that do not change with time. Electromagnetic field carries energy-momentum [2]

$$p_{\rm em}^{\nu}=\int_{\Sigma}d\sigma_\alpha T^{\alpha\nu} \tag{1.2}$$

and angular momentum

$$M_{\rm em}^{\mu\nu}=\int_{\Sigma}d\sigma_\alpha\left(y^{\mu}T^{\alpha\nu}-y^{\nu}T^{\alpha\mu}\right) \tag{1.3}$$

which flow across a space-like surface Σ with vectorial surface element $d\sigma_\alpha$. Since the Maxwell energy-momentum tensor density

$$4\pi T^{\mu\nu}=f^{\mu\lambda}f^{\nu}_{\ \lambda}-1/4\eta^{\mu\nu}f^{\kappa\lambda}f_{\kappa\lambda} \tag{1.4}$$

is quadratic in the field and this field satisfies the superposition principle, the total electromagnetic field stress-energy tensor is

$$T^{\mu\nu}=T_{(1)}^{\mu\nu}+T_{(2)}^{\mu\nu}+T_{\rm int}^{\mu\nu} \tag{1.5}$$

where a-th particle density $T_{(a)}^{\mu\nu}$ is given by the expression (1.4) where "total" field strengths $f^{\mu\nu}$ are substituted by "individual" ones $f_{(a)}^{\mu\nu}$. The mixed term

$$4\pi T_{\rm int}^{\mu\nu}=f_{(1)}^{\mu\lambda}f^{\nu}_{(2)\lambda}+f_{(2)}^{\mu\lambda}f^{\nu}_{(1)\lambda}-1/4\eta^{\mu\nu}\left(f_{(1)}^{\kappa\lambda}f^{(2)}_{\kappa\lambda}+f_{(2)}^{\kappa\lambda}f^{(1)}_{\kappa\lambda}\right) \tag{1.6}$$

describes the joint contribution due to both fields.

The contributions to Noether quantities due to "individual" parts $\hat T_{(1)}$ and $\hat T_{(2)}$ of the Maxwell tensor density (1.5) are the same as in one-particle problem. The bound and emitted energy-momentum is calculated in [3], the angular momentum carried by the electromagnetic field of an accelerated point charge is obtained in [4]. The authors [3,4] deal only with the retarded Liénard-Wiechert solution; the advanced one is rejected on the grounds of causality. Following the method originally developed by Dirac [1], they evaluate the energy-momentum [3] and angular momentum [4] emitted by segment of the particle's world line across a narrow tube the radius of which will in the end by made to tend to zero. The tube is ended by two tilted hyperplanes; each of them is orthogonal to the particle's 4-velocity referred to the corresponding end point of the segment.

Teitelboim [3] decomposes the stress-energy tensor into two divergent-free components: $\hat{T} = \hat{T}_{\rm bnd} + \hat{T}_{\rm rad}$. The surface integration of the bound part $\hat{T}_{\rm bnd}$ results the terms which are permanently "attached" to the charge and are carried along with it[1]. The integration of the radiative component $\hat{T}_{\rm rad}$ yields the Larmor relativistic rate of radiated energy-momentum. This part of energy-momentum detaches itself from the charge and leads an independent existence. Similarly, López and Villarroel split the torque of the stress-energy tensor into bound and radiative components which possess analogous properties. Unavoidable infinities stemming from the pointness of the source are absorbed by particle's individual characteristics within the renormalization procedure while the finite radiative parts of Noether quantities exert the radiation reaction.

The difficulties associated with the computation of the mixed contribution (1.6) are twofold – to perform the meaningful decomposition of $\hat{T}_{\rm int}$ into bound and radiative parts and to choose an appropriate surface of integration. The tilted hyperplane which plays privileged role in the one-particle radiation reaction problem [3,4] is not suitable whenever two-body one is considered. Indeed, there is no a hyperplane which is orthogonal to the world lines of both the particles at all events. In Ref. [5] the fundamental theorem is proven that the mixed *radiation* rate does not depend on the shape of space-like surface which is used to integrate $\hat{T}_{\rm int}$[2]. It is convenient to choose the hyperplane $\Sigma_t = \left\{ y \in \mathcal{M}_4 : y^0 = t \right\}$ associated with an unmoving inertial observer. The "laboratory" time t is a single common parameter defined along all the world lines of our two-body system. Having performed the surface integration, we reveal the radiative component of $\hat{T}_{\rm int}$ which produces finite manifestly covariant terms which constitute mixed contribution to radiated energy-momentum.

The present paper is organized as follows. In Section 2 we present the integrations of one-particle stress-energy tensor $\hat{T}_{(a)}$ and its torque which are patterned after Teitelboim's classic paper [3], although they differ from it in their technical aspects. In Section 3 we trace a series of stages in the integration of the mixed term which is due to combination of outgoing electromagnetic waves of the first charge and the second charge. In Section 4 we analyse conservation laws corresponding to Poincaré symmetry of our closed particles'

[1]The divergency which is proportional to the charge's self-energy and the well-known Schott term arise.
[2]The form of bound terms which describe the deformation of electromagnetic "clouds" of "bare" charges due to mutual interaction depend on the shape of Σ only.

plus field system. In Appendix we present the radiative component $\hat{T}_{\rm int,rad}$ of mixed stress-energy tensor (1.6) which produces the interference part of emitted energy-momentum. In Section 5 we discuss the results and their implications.

2. One-particle radiation reaction problem

In this Section we compute the contributions to Noether quantities due to "individual" parts $\hat{T}_{(1)}$ and $\hat{T}_{(2)}$ of the electromagnetic field's stress-energy tensor (1.5). We calculate the energy-momentum and angular momentum produced by all points of the world line $\zeta_a : \mathcal{R} \to \mathcal{M}_4$ up to the end point at which ζ_a punctures the hyperplane Σ_t. We enclose a-th world line by Bhabha tube [6] of a constant radius r which is not necessarily small. The tube is the disjoint union of spheres $S(z_a(s_a),r)$ where the evolution parameter s_a indicates the portion of the world line that corresponds to the interval $-\infty < s_a \le s_a^e$. The sphere is the intersection of the future light cone generated by null rays emanating from $z_a(s_a) \in \zeta_a$ in all possible directions

$$C(z_a(s_a)) = \left\{ y \in \mathcal{M}_4 : \eta_{\alpha\beta}\left(y^\alpha - z_a^\alpha(s_a)\right)\left(y^\beta - z_a^\beta(s_a)\right) = 0,\ y^0 - z_a^0(s_a) > 0 \right\} \qquad (2.1)$$

and the tilted hyperplane

$$\sigma(z_a(s_a),r) = \left\{ y \in \mathcal{M}_4 : u_{a\,\alpha}(s_a)\left(y^\alpha - z_a^\alpha(s_a) - u_a^\alpha(s_a) r\right) = 0 \right\} \qquad (2.2)$$

The parameter s_a is called the retarded time, it is the root of algebraical equation (2.1) which satisfies the causality condition $y^0 > z_a^0(s_a)$. The instant s_a^e labels the vertex E of the forward light cone such that r-shifted hyperplane (2.2) contains the end point $P = \zeta_a \cap \Sigma_t$ with coordinates (t, $\mathbf{z}_a$ (t)) (see Figure 1).

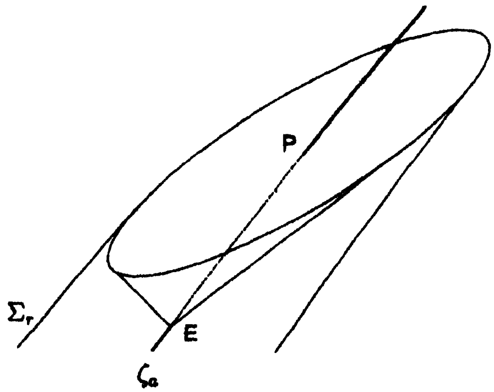

Figure 1. The grey oval pictures the cap of the Bhabha tube Σ_r. It is the fragment of r-shifted hyperplane $\sigma(E,r)$ which is orthogonal to particle's 4-velocity at point E. Walls of the Bhabha tube are built by spheres of type $S(E,r) = C(E) \cap \sigma(E,r)$ where $C(E)$ is the forward light cone with vertex at point of emission. At point P the world line ζ_a punctures the cap

To distinguish points on the Bhabha tube Σ_r we introduce curvilinear coordinates $(s_a, r, \vartheta, \phi)$ locally given by

$$y^\alpha = z_a^\alpha(s_a) + r\,k_a^\alpha \tag{2.3}$$

This coordinate transformation belongs to the class of retarded coordinate system described by Newman and Unti in [7] We suppose that null vector k^α is equal to $\Lambda^\alpha{}_\alpha n^\alpha$ where the matrix Λ passes to the particle's momentarily comoving Lorentz frame (MCLF) where a-th particle is momentarily at rest at the retarded instant s_a(see Figure 2) The null vector n has the components $(1, \cos\phi\sin\vartheta, \sin\phi\sin\vartheta, \cos\vartheta)$ Spherical polar angles ϕ and ϑ parametrize points on the sphere $S(z_a(s_a), r) = C(z_a(s_a)) \cap \sigma(z_a(s_a), r)$ which is the intersection of the future light cone (2 1) and r-shifted hyperplane (2 2)

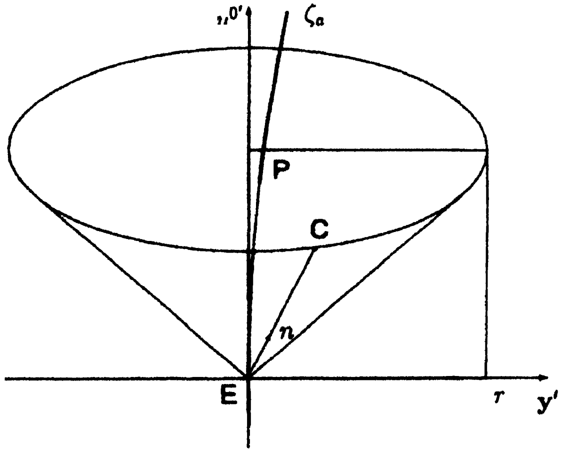

Figure 2 In MCLF the charge is placed in the coordinate origin, it is momentarily at the rest The point C on spherical light front S (0, r) is linked to the coordinate origin by a null ray characterized by the angles ϑ and ϕ specifying its direction on the cone The vector n is tangent to this null ray

In terms of retarded coordinates (2 3) the electromagnetic field generated by a-th particle is given by

$$\hat{f}_{(a)} = \frac{e_a}{r^2} u_a \Lambda k_a + \frac{e_a}{r}\left[a_a \Lambda\, k_a + (k_a\ a_a) u_a\, \Lambda\, k_a\right] \tag{2.4}$$

where the symbol Λ denotes the wedge product It is straightforward to substitute these components into expression (1 4) to calculate the stress-energy tensor $\hat{T}_{(a)}$ Following [3], we present it as a sum of bound and radiative components The former is the combination of terms which depend on the radius of tube as r^{-4} and r^{-3}

$$T^{\mu\nu}_{(a)\,\mathrm{bnd}} = \frac{e_a^2}{4\pi r^4}\left[2k_a^\mu u_a^\nu + 2k_a^\nu u_a^\mu - k_a^\mu k_a^\nu - \frac{1}{2}\eta^{\mu\nu}\right.$$

$$+\frac{e_a^2}{2\pi r^3}\left[k_a^\mu a_a^\nu + k_a^\nu a_a^\mu + (a_a\cdot k_a)\left(k_a^\mu u_a^\nu + k_a^\nu u_a^\mu - k_a^\mu k_a^\nu\right)\right] \tag{2.5}$$

while the latter scales as r^{-2}

$$T^{\mu\nu}_{(a),\mathrm{rad}} = \frac{e_a^2}{4\pi r^4}\left[a_a^2 - (a_a\cdot k_a)^2\right]k_a^\mu k_a^\nu \tag{2.6}$$

The torque $y^\mu T^{\alpha\nu}_{(a)} - y^\nu T^{\alpha\mu}_{(a)}$ also can be decomposed into the bound and radiative components [4, eqs (2.10)].

We calculate the flows of energy-momentum and angular momentum which flow across the walls of the Bhabha tube. According to [6], the outward directed surface element $d\sigma_\mu$ of the cylinder $r = const$ is

$$d\sigma_\mu = \left[-u_{a\,\mu} + (1 + ra_k)k_{a\,\mu}\right] r^2 d\Omega ds_a \tag{2.7}$$

where $a_k := (a_a\cdot k_a)$ is the component of particle's acceleration in the direction k_a and $d\Omega = \sin\vartheta d\vartheta d\phi$ is the element of a solid angle.

Inserting $\hat{T}_{(a)}$ and its torque $y^\mu T^{\alpha\nu}_{(a)} - y^\nu T^{\alpha\mu}_{(a)}$ into eq. (1 2) and eq (1.3), respectively, and integrating over the angular variables associated with the vector k_a yields

$$p^\nu_{\mathrm{wall}}\left(s_a^e\right) = p^\nu_{\mathrm{bnd}}\left(s_a^e\right) + p^\nu_{\mathrm{rad}}\left(s_a^e\right) \tag{2.8}$$

$$\frac{e_a^2}{2r}u_a^\nu(s_a)\Bigg|_{s_a\to-\infty}^{s_a=s_a^e} + \frac{2e_a^2}{3}\int_{-\infty}^{s_a^e} ds_a a_a^2(s_a)u_a^\nu(s_a)$$

$$M^{\mu\nu}_{\mathrm{wall}}(s_a^e) = M^{\mu\nu}_{\mathrm{bnd}}\left(s_a^e\right) + M^{\mu\nu}_{\mathrm{rad}}\left(s_a^e\right) \tag{2.9}$$

$$= \frac{e_a^2}{2r}\left[z_a^\mu(s_a)u_a^\nu(s_a) - z_a^\nu(s_a)u_a^\mu(s_a)\right]\Bigg|$$

$$+\frac{2e_a^2}{3}\int_{-\infty}^{s_a^e} ds_a\left\{a_a^2(s_a)\left[z_a^\mu u_a^\nu - z_a^\nu u_a^\mu\right] + u_a^\mu a_a^\nu - u_a^\nu a_a^\mu\right\}.$$

The bound terms depend on the state of the particle's motion at the remote past and at the moment s_a^e which determines the edge of the Bhabha tube while the radiative ones are accumulated with time.

We are now concerned with the calculation of the flows of energy-momentum (1 2) and angular momentum (1 3) through the cap of the Bhabha tube It is the fragment of the tilted hyperplane $\sigma\left(z_a(s^e_a), r\right)$ which is orthogonal to the particle's 4-velocity at point E The surface is given by eq (2 2) where $s_a = s^e_a$ The domain of integration corresponds to the segment of the world line between the point E with coordinates $z_a(s^e_a)$ and the point P with coordinates (t, $\mathbf{z}_a$ (t)) (see Figure 1)

We make the Lorentz transformation associated with the state of particle's motion at point E After that the tilted hyperplane $\sigma\left(z_a(s^e_a), r\right)$ becomes $\Sigma_t = \left\{y' \in \mathcal{M}_4 \;\; y^0 = r\right\}$ as it is pictured in Figure 2 On rearrangement, energy-momentum (1 2) and angular momentum (1 3) take the form

$$p^{\prime}_{a\,\mathrm{cap}}(t) = \Lambda^{\prime}{}_{\iota}(s^e_a)\int_{y^0=r} d\sigma_0\, T^{0\iota}_{(a)} \tag{2 10}$$

$$M^{\mu\iota}_{a\,\mathrm{cap}}(t) = \Lambda^{\mu}{}_{\mu}(s^e_a)\Lambda^{\iota}{}_{\iota}(s^e_a)\int_{y^0=r} d\sigma_0 \left(y^{\mu}T^{0\iota}_{(a)} - y^{\iota}T^{0\mu}_{(a)}\right) \tag{2 11}$$

To adopt curvilinear coordinates (2 3) to an integration surface $\Sigma_t = \left\{y \in \mathcal{M}_4 \;\; y^0 = t\right\}$, we replace the radius r by the expression

$$r = \frac{t-\lambda}{} \tag{2 12}$$

where t is the observation time and λ is the evolution parameter which parametrize the segment EP of ζ_a We arrive at the following coordinate transformation $\left(y^{\alpha}\right) \to (t, \lambda, \vartheta, \phi)$

$$y^0 = t, \quad y^{\prime} = z^{\prime}_a(\lambda) + \frac{t-\lambda}{k^0}k^{\prime}_a \tag{2 13}$$

If parametrization of the world line is provided by a disjoint union of hyperplanes $\Sigma_\lambda = \left\{y \in \mathcal{M}_4 \;\; y^0 = \lambda\right\}$, particle's velocity takes the form $u^{\mu}_a = \gamma_a \upsilon^{\mu}_a$, $\upsilon^{\mu}_a = \left(1, \dot{z}^{\prime}_a\right)$, and acceleration $a^{\mu}_a = \gamma^4_a(\mathbf{v}_a\dot{\mathbf{v}}_a)\upsilon^{\mu}_a + \gamma^2_a\dot{\upsilon}^{\mu}_a$, factor $\gamma_a = 1/\sqrt{1-\mathbf{v}^2_a}$ (The overdot indicates differentiation with respect to λ)

New surface element is as follows

$$d\sigma_0 = \sqrt{1-\mathbf{v}^2_a}\,\frac{(1-\lambda)^2}{\left(k^0_a\right)^3}\,d\lambda\, d\Omega \tag{2 14}$$

Inserting this and (2.5) into eq. (1.2) and integrating $\hat{T}_{\rm bnd}$ over the angular variables yields

$$p^0_{a,\rm bnd}(\text{cap}) = \frac{2e_a^2}{3}\int_{t-r}^{t} d\lambda \left\{\frac{1}{(t-\lambda)^2}\left(-\frac{1}{4}+\frac{1}{1-\mathbf{v}_a^2}\right) + \frac{1}{t-\lambda}\frac{2(\mathbf{v}_a\dot{\mathbf{v}}_a)}{\left(1-\mathbf{v}_a^2\right)}\right\}$$

$$-\frac{2e_a^2}{3}\left.\frac{1}{t-\lambda}\left(-\frac{1}{4}+\frac{1}{1-\mathbf{v}_a^2(\lambda)}\right)\right|_{\lambda=t-4} \qquad (2\ 15)$$

$$p^i_{a,\rm bnd}(\text{cap}) = \frac{2e_a^2}{3}\int_{t-r}^{t} d\lambda \left\{\frac{1}{(t-\lambda)^2}\frac{\upsilon_a^i}{1-\mathbf{v}_a^2} + \frac{1}{t-\lambda}\left[\frac{\dot{\upsilon}_a^i}{1-\mathbf{v}_a^2}+\frac{2(\mathbf{v}_a\dot{\mathbf{v}}_a)\upsilon_a^i}{\left(1-\mathbf{v}_a^2\right)^2}\right]\right\}$$

$$-\frac{2e_a^2}{3}\left.\frac{1}{t-\lambda}\frac{\upsilon_a^i(\lambda)}{1-\mathbf{v}_a^2(\lambda)}\right|_{\lambda=t-4}^{\lambda\to t} \qquad (2\ 16)$$

In analogous way we calculate the bound part of angular momentum tensor which also depends on the state of the source at the end points :

$$M^{0i}_{a,\rm bnd}(\text{cap}) = \frac{2e_a^2}{3}\left[\frac{\lambda}{t-\lambda}\frac{\upsilon_a^i(\lambda)}{1-\mathbf{v}_a^2(\lambda)} - \frac{z_a^i(\lambda)}{t-\lambda}\left(\frac{1}{1-\mathbf{v}_a^2(\lambda)}-\frac{1}{4}\right)\right]^{\lambda\to t} \qquad (2\ 17)$$

$$M^{ij}_{a,\rm bnd}(\text{cap}) = \frac{2e_a^2}{3}\left[\frac{1}{t-\lambda}\frac{z_a^i(\lambda)\,\upsilon_a^j(\lambda)-z_a^j(\lambda)\,\upsilon_a^i(\lambda)}{1-\mathbf{v}_a^2(\lambda)}\right]_{\lambda=t-r}^{\lambda\to t} \qquad (2.18)$$

Following eqs. (2.10) and (2.11), we pass to the momentarily comoving Lorentz frame associated with the state of particle's motion at point *E*. If $v^i\big|_{t-r}=0$ the lower limits in eqs. (2.15)–(2.18), coincide exactly with the upper limits of the bound energy-momentum (2.8) and angular momentum (2.9) taken at MCLF where $u_a^\nu\left(s_a^\bullet\right)=(1,0,0,0)$. We see that the solutions are sewn at the edge of the Bhabha tube The upper limits diverge in eqs. (2.15)–(2.18) as could be expected for the bound parts of Noether quantities evaluated in the immediate vicinity of the world line.

The radiation parts of the electromagnetic field's energy-momentum and angular momentum do not depend on choosing of an integration surface :

$$M^{\nu}_{a,\mathrm{rad}}(\mathrm{cap}) = \frac{2e_a^2}{3} \int_{t-r}^{t} d\lambda \gamma_a^{-1} \; a_a^2(\lambda) \; u_a^{\nu}(\lambda) \tag{2.19}$$

$$M^{4\nu}_{a,\mathrm{rad}}(\mathrm{cap}) = \frac{2e_a^2}{3} \int_{t-r}^{t} d\lambda \gamma_a^{-1} \left\{ a_a^2(s_a) \left[z_a^{\mu} u_a^{\nu} - z_a^{\nu} u_a^{\mu} \right] + u_a^{\mu} a_a^{\nu} - u_a^{\nu} a_a^{\mu} \right\}. \tag{2.20}$$

Passing to the proper time $ds_a = d\lambda\gamma_a^{-1}$, we rewrite them in a manifestly covariant fashion. New expressions look as the radiative terms in eqs. (2.8) and (2.9). The domain of integration corresponds to the segment of the world line between the points E and P (see Figures 1 and 2).

We do not derive the Schott term from the bound part of the Maxwell energy-momentum tensor density. To find the desired expression we will analyse conserved quantities corresponding to the Poincaré invariance of the theory. In balance equations we will take into account the radiative parts of Noether quantities carried by electromagnetic field only. The conservation laws are an immovable fulcrum about which tips the balance of truth regarding renormalization and radiation reaction.

3. Joint contribution to Noether quantities

In Section 2 we compute the contributions to Noether quantities due to "individual" parts $\hat{T}_{(1)}$ and $\hat{T}_{(2)}$ of the electromagnetic field's stress-energy tensor (1.5). In this Section we evaluate the joint contribution due to both fields corresponding to the mixed part (1.6) of Maxwell's tensor.

Having considered the one-particle radiation reaction problem we enclose particle's world line by Bhabha tube [6] of a constant radius r which is not necessarily small. To avoid the loss of radiation we furnish the tube by a "cap". We calculate the flows of energy-momentum and angular momentum which flow across both the wall and the "cap" of the tube. We rederive the well-known Larmor expression for emitted energy-momentum and the rate of radiated angular momentum which arise in the one-particle self-action problem [3,4].

If two-body problem is considered we should enlarge the tube as much as possible to enclose a region of interaction. In order to take proper account of radiation emanated by charges at remote past we tend its radius to infinity. The radiation does not reach wall of the tube at all and flows across an unbounded "cap". As a "cap" we choose the simplest hyperplane $\Sigma_t = \left\{ y \in \mathcal{M}_4 : y^0 = t \right\}$ associated with an unmoving inertial observer.

To reveal meaningful radiative parts of energy-momentum

$$p^{\nu}_{\mathrm{int}}(t) = \int_{\Sigma t} d\sigma_0 \; T^{0\nu}_{\mathrm{int}} \tag{3.1}$$

and angular momentum

$$M_{\rm int}^{\mu\nu}(t) = \int_{\Sigma_t} d\sigma_0 \left(y^\mu T_{\rm int}^{0\nu} - y^\nu T_{\rm int}^{0\mu} \right) \qquad (3.2)$$

we apply the criteria which were first formulated in [3, Table 1] :

- the bound term diverges while the radiative one is finite;
- the bound component depends on the momentary state of the particles' motion while the radiative one is accumulated with time; and
- the form of the bound terms heavily depends on choosing of an integration surface while the radiative terms are invariant.

3.1. Coordinate system :

An appropriate coordinate system for integration eqs. (3.1) and (3.2) of the mixed quantities over tilted hyperplane associated with inertial observer is introduced by Aguirregabiria and Bel in [5] (see also [8]). In Refs. [9,10] this system is adapted to the hyperplane Σ_t. Flat spacetime is spanned by curvilinear coordinates (t, t_1, t_2, φ) presented in Figure 3.

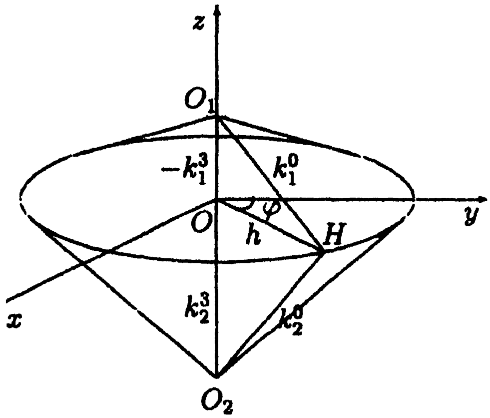

Figure 3. The sphere $S_1(O_1, t-t_1)$ with centre at point O_1 and radius $k_1^0 = t - t_1$ is the intersection of the future light cone with vertex at point $z_1(t_1) \in \zeta_1$ and Σ_t. The sphere $S_2(O_2, t-t_2)$ centred at point O_2 is the intersection of Σ_t and the forward light cone of $z_2(t_2) \in \zeta_2$. Polar angle φ distinguishes the points of their intersection which constitute support of integrals (3 1) and (3.2). k_a^0, k_a^3 and h are the components of the future oriented null 4-vector $k_a^{\alpha'} = \Omega^{\alpha'}{}_{\alpha}\left(y^\alpha - z_a^\alpha(t_a)\right)$ where $\hat\Omega$ is orthogonal matrix. It determines transition to "momentarily rotating" Lorentz frame where z-axis is directed along 3-vector $\mathbf{q} := \mathbf{z}_1 - \mathbf{z}_2$

The mixed contribution is due to interference of spherical wave fronts S_1 and S_2 in Σ_t which is pictured in Figures 3 and 4. Clearly, this contibution is zero if the relative 4-vector $q = z_1 - z_2$ is timelike. If the 4-vector is spacelike, the intersection $S_1 \cap S_2$ becomes the circle $C(O, h)$ with radius h; in "momentarily rotating" Lorentz frame it lies of Oxy plane and centred at the coordinate origin. If points z_1 and z_2 are related by a null ray, the intersection $S_1 \cap S_2$ contains the only point, either N or S (see Figure 4).

The integration of mixed part of the stress-energy tensor (3.1) and its torque (3.2) is very cumbersome. For this reason the details will be presented elsewhere. It is of crucial issue the integration over φ results the combination of partial derivatives in times t_1 and t_2. Further integration over time variables results functions of the limits of integrals, $t_b^{\rm ret}(t_a)$ and $t_b^{\rm adv}(t_a)$, which mark instants such that corresponding wave fronts touch each other at only point. For this reason the retarded and the advanced Liénard-Wiechert solutions arise. And yet the retarded causality is not violated. Indeed, all the moments are *before* the observation instant t (see Figure 4).

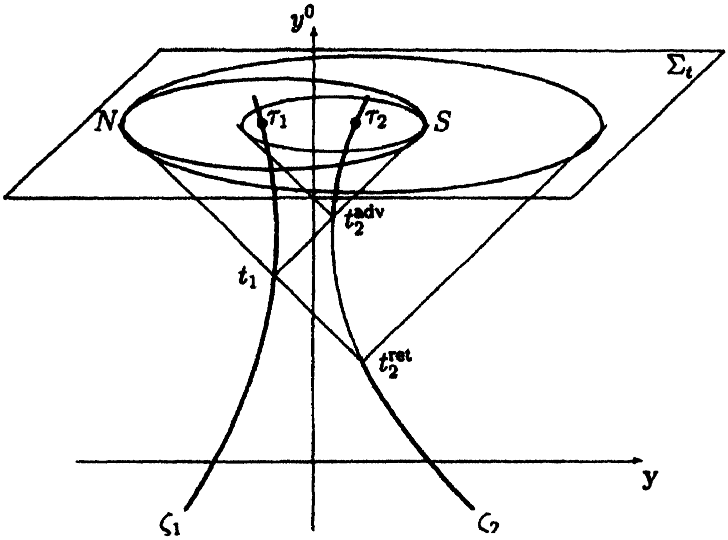

Figure 4. For a given t_1 the wave front S_1 $(z_1(t_1))$ is covered by circles $S_1 \cap S_2$ if the parameter t_2 increases from $t_2^{\rm ret}(t_1)$ to $t_2^{\rm adv}(t_1)$. Minimal value lables the vertex of forward light cone which is punctured by the world line of the first charge at a given point $z_1(t_1)$. Sphere $S_2\left(z_2^{\rm ret}\right)$ touches the given sphere S_1 at point N. The world line of the second charge punctures the future light cone of $z_1(t_1)$) at point $z_2\left(t_2^{\rm adv}\right)$. Intersection $S_1 \cap S_2^{\rm adv}$ contains the only point S.

3.2. *Mixed parts of energy-momentum and angular momentum* :

Although "interference" coordinate system is adopted to the non-covariant hyperplane Σ_t, the joint contribution to radiative parts of Noether quantities is manifestly covariant. Radiative part of mixed energy-momentum carried by electromagnetic field of two point-like source involves the retarded and the advanced Liénard-Wiechert fields :

$$
p_{\rm int}^{\rm rad}(t) = -\frac{1}{2}\left[\int_{-\infty}^{\tau_1} ds_1\, F_{21}^{\rm ret} + \int_{-\infty}^{\tau_1^{\rm ret}} ds_1\, F_{21}^{\rm adv}\right] - \frac{e_1}{2}\left(A_{21}^{\rm ret}\Big|_{-\infty}^{\tau_1} + A_{21}^{\rm adv}\Big|_{-\infty}^{\tau_1^{\rm ret}}\right.
$$

$$
\left. -\frac{1}{2}\left[\int_{-\infty}^{\tau_2} ds_2\, F_{12}^{\rm ret} + \int_{-\infty}^{\tau_2^{\rm ret}} ds_2\, F_{12}^{\rm adv}\right] - \frac{e_2}{2}\left(A_{12}^{\rm ret}\Big|_{-\infty}^{\tau_2} + A_{12}^{\rm adv}\Big|_{-\infty}^{\tau_2^{\rm ret}}\right). \tag{3.3}
$$

The retarded (advanced) Lorentz force F_{ba} is the convolution of 4-velocity $u_a(s_a)$ and the retarded (advanced) electromagnetic field generated by b-th particle at point where a-th particle is located (Charges a and b interact if and only if they are connectible by a null ray.) Limits of integrals τ_1 and τ_2 are the instants which indicate the points at which the world lines puncture the observation hyperplane Σ_t (see Figure 4). The moment $\tau_b^{\rm ret}(\tau_a)$ labels the point on ζ_b such that $z_a(\tau_a) \in \zeta_a$ and $z_b\left(\tau_b^{\rm ret}\right) \in \zeta_b$ are linked by a null ray

By large letter A we denote the Liénard-Wiechert potentials referred to the limits of integrals. At the upper limits we have

$$A_{12}^{\rm ret} = e_1 \frac{u_1\left(\tau_1^{\rm ret}\right)}{\left(q \cdot u_1^{\rm ret}\right)}, \quad A_{21}^{\rm ret} = e_2 \frac{u_2\left(\tau_2^{\rm ret}\right)}{-\left(q \cdot u_2^{\rm ret}\right)} \tag{3.4}$$

$$A_{12}^{\rm adv} = e_1 \frac{u_1(\tau_1)}{(q \cdot u_1)}, \quad A_{21}^{\rm adv} = e_2 \frac{u_2(\tau_2)}{-(q \cdot u_2)}$$

Radiative part of interference angular momentum carried by two-body electromagnetic field also contains both the retarded and the advanced solutions of Maxwell equations ·

$$\hat{M}_{\rm int}^{\rm rad}(t) = -\frac{1}{2}\left[\int_{-\infty}^{\tau_1} ds_1 z_1 \wedge F_{21}^{\rm ret} + \int_{-\infty}^{\tau_1^{\rm ret}} ds_1 z_1 \wedge F_{21}^{\rm adv}\right] \tag{3.5}$$

$$-\frac{e_1}{2}\left(z_1 \wedge A_{21}^{\rm ret}\Big|_{-\infty}^{\tau_1} + z_1 \wedge A_{21}^{\rm adv}\Big|_{-\infty}^{\tau_1^{\rm ret}}\right.$$

$$-\frac{1}{2}\left[\int_{-\infty}^{\tau_2} ds_2 z_2 \wedge F_{12}^{\rm ret} + \int_{-\infty}^{\tau_2^{\rm ret}} ds_2 z_2 \wedge F^{\rm adv}\right.$$

$$-\frac{e_2}{2}\left(z_2 \wedge A_{12}^{\rm ret}\Big|_{-\infty}^{\tau_2} + z_2 \wedge A_{12}^{\rm adv}\right.$$

We see that the surface integration reduces uncountably infinite degrees of freedom Contrary to the electromagnetic field in the Maxwellian field theory originated from the variation of action (1.1), the fields in expressions (3.3) and (3.5) do not have degrees of freedom of their own. They are functionals of particles' pathes as it is in action at a distance electrodynamics developed by Wheeler and Feynman in their classic papers [11, 12]. (For a modern review see [13].)

Action at a distance electrodynamics satisfies the postulational requirement of the complete time symmetry. Wheeler and Feynman assume that the fields which act on a given particle from other particles are represented by one-half the retarded plus one-half the advanced Liénard-Wiechert solutions. It means that the electromagnetic interactions proceeds not only forward in time but also backwards in time. But the expressions (3.3) and (3.5) do not contain the waves *converging* from infinity onto the source. They summarize the study of interference of *outgoing* waves in the observation hyperplane Σ_t. To make the expressions time symmetric we locate the observation hyperplane $y^0 = t$ in the distant future. We assume that particles are asymptotically free at the remote past and at the distant future. The local terms vanish[3] and radiated energy-momentum (3.3) becomes invariant with respect to the time inversion :

$$p_{\text{int}}^{\text{rad}} = -\frac{1}{2}\int_{-\infty}^{+\infty} ds_1\left(F_{21}^{\text{ret}} + F_{21}^{\text{adv}}\right) - \frac{1}{2}\int_{-\infty}^{+\infty} ds_2\left(F_{12}^{\text{ret}} + F_{12}^{\text{adv}}\right). \tag{3.6}$$

Indeed, the retarded Lorentz force, F_{ba}^{ret}, becomes the advanced one, F_{ba}^{adv}, (and *vice versa*) if the time direction is reversed.

If $t \to +\infty$ the "interference" angular momentum (3.5) also looks be symmetric with respect to past and future :

$$M_i^{\text{rad}} = -\frac{1}{2}\int_{-\infty}^{+\infty} ds_1 z_1 \wedge \left(F_{21}^{\text{ret}} + F_{21}^{\text{adv}}\right) - \frac{1}{2}\int_{-\infty}^{+\infty} ds_2 z_2 \wedge \left(F_{12}^{\text{ret}} + F_{12}^{\text{adv}}\right). \tag{3.7}$$

However, complete reversibility is not achieved because the radiative damping caused by "individual" contributions to the energy-momentum and angular momentum (see Section 2) In Wheeler and Feynman electrodynamics the terms describing radiative damping arise if the assumption of "complete absorption" is applied to the particles' motion equations.

3.3. Perfect absorber :

In the action at a distance theory [11,12] the so-called "perfect absorber" cancels the one-half of advanced force acted on a given particle from other ones and doubles the one-half of retarded force. The absorber consists of all charges of Univese because the pointed charge radiates at all possible directions. According to [12, pg. 427], "In a universe consisting of a limited number of charged particles advanced effects occur explicitly."

In this subsection we present relationships between the advanced terms in eqs. (3.3) and (3.5) and their retarded counterparts. We do not suppose the electromagnetic interactions can proceed backward in time. We still consider the interference of outgoing electromagnetic waves at the observation hyperplane Σ_t. Figure 4 shows that the retarded and the *advanced* instants arise naturally when the combination of *outgoing* waves is studied.

The crucial issue is that the functions $s_a^{\text{adv}}(s_b)$ and $s_b^{\text{ret}}(s_a)$ are inverses. It allows us to change the variables $\left(s_a^{\text{adv}}, s_b\right) \to \left(s_b^{\text{ret}}, s_a\right)$ in the "advanced" integrals of eqs. (3.3)

[3] Liénard-Wiechert potentials fall off at large distances inversely as the first power of the separation vector $q = z_1 - z_2$ between the charges.

and (3.5) An essential feature of integration is that the difference between the work done by the retarded Lorentz force due to charge b on charge a and the work done by "advanced" response due to charge a on charge b constitute the integral being a function of the end points only

$$\int_{-\infty}^{\tau_a} ds_a F_{ba}^{\rm ret} - \int_{-\infty}^{\tau_b^{\rm ret}} ds_b F_{ab}^{\rm adv} = \left[e_b A_{ab}^{\rm adv} - e_a A_{ba}^{\rm ret} + (-1)^b q\left(A_{ab}^{\rm adv}\cdot A_{ba}^{\rm ret}\right)\right]_{s_a\to-\infty}^{s_a=\tau_a} \tag{3 8}$$

Taking the limit $t\to+\infty$ we restore time-reversal invariance. Namely, the work done by retarded Lorenz force of b-th charge over entire world line of a-th one is equal to the work done by advanced Lorentz force of a-th particle acting on b-th charge backward in time

$$\int_{-\infty}^{+\infty} ds_a F_{ba}^{\rm ret} = \int_{-\infty}^{+\infty} ds_b F_{ab}^{\rm adv}. \tag{3 9}$$

The first charge seems a "perfect absorber" for the advanced radiation given off by the second one and *vice versa*

Inserting the relation (3 8) in eq (3 3) we cancel the advanced terms and double the retarded ones

$$p_{\rm int}^{\rm rad}(t) = -\int_{-\infty}^{\tau_1} ds_1 F_{21}^{\rm ret} - \left[e_1 A_{21}^{\rm ret} - \frac{1}{2}d_{21}\right]_{s_1\to-\infty}^{s_1=\tau_1} \tag{3 10}$$

$$-\int_{-\infty}^{\tau_1} ds_2 F_{12}^{\rm ret} - \left[e_2 A_{12}^{\rm ret} - \frac{1}{2}d_{12}\right]_{s_2\to-\infty}^{s_2=\tau_2}$$

By d_{ab} we denote q-directed 4-vector which is proportional to the scalar product of Liénard-Wiechert potentials :

$$d_{ba} = (-1)^b q\left(A_{ba}^{\rm ret}\cdot A_{ab}^{\rm adv}\right) \tag{3 11}$$

If the upper limits are considered the Liénard-Wiechert potentials are given by eqs. (3 4)

Difference between the integrals of the torque retarded force and its advanced counterpart looks as follows :

$$\int_{-\infty}^{\tau_1} ds_a z_a \wedge F_{ba}^{\rm ret} - \int_{-\infty}^{\tau_b^{\rm ret}} ds_b z_b \wedge F_{ab}^{\rm adv} \tag{3.12}$$

$$= \left[e_b z_b \wedge A_{ab}^{\rm adv} - e_a z_a \wedge A_{ba}^{\rm ret} + z_a \wedge d_{ba}\right]_{s_a\to-\infty}^{s_a=\tau_a}$$

If $t\to+\infty$ the relation becomes invariant with respect to time reflection

Inserting this in eq. (3.5) we remove the advanced integrals :

$$\hat{M}^{\text{rad}}_{\text{int}}(t) = -\int_{-\infty}^{\tau_1} ds_1 z_1 \wedge F^{\text{ret}}_{21} - \left[z_1 \wedge \left(e_1 A^{\text{ret}}_{21} - \frac{1}{2} d_{21} \right) \right]_{-\infty}^{\tau_1} \tag{3 13}$$

$$-\int_{-\infty}^{\tau_2} ds_2 z_2 \wedge F^{\text{ret}}_{12} - \quad z_2 \wedge \quad e_2 A^{\text{ret}}_{12} - \frac{1}{2} d_{12}$$

4. Equations of motion of radiating charges

Summing up the individual and the joint contributions, we introduce the radiative part of energy-momentum

$$p^{\text{rad}}(t) = \sum_{a=1}^{2} \frac{2e_a^2}{3} \int_{-\infty}^{\tau_a} ds_a a_a^2(s_a) u_a(s_a) + p^{\text{rad}}_{\text{int}}(t) \tag{4.1}$$

and angular momentum

$$\hat{M}^{\text{rad}}(t) = \sum_{a=1}^{2} \frac{2e_a^2}{3} \int_{-\infty}^{\tau_a} ds_a \left[a_a^2(s_a) z_a \wedge u_a + u_a \wedge a_a \right] + \hat{M}^{\text{rad}}_{\text{int}}(t) \tag{4 2}$$

carried by electromagnetic field of two sources acting on one another through the medium of the retarded Liénard-Wiechert field. These quantities together with the sum of particles' individual 4-momenta and angular momenta constitute the total conserved quantities of our closed particles plus field system. The change in field's energy-momentum and angular momentum should be balanced by a corresponding change of particles' 4-momenta and angular momenta, respectively. Since the action is not propagated instantaneously, the balance in a vicinity of the first charge as well as in a neighbourhood of the second charge should be achieved separately :

$$\dot{p}_{\text{part},a}(\tau_a) = -\frac{2e_a^2}{3} a_a^2(\tau_a) u_a + F^{\text{ret}}_{ba} + e_a \dot{A}^{\text{ret}}_{ba} - \frac{1}{2} \dot{d}_{ba}. \tag{4.3}$$

(The overdot means the derivation with respect to individual proper time τ_a.) Index a indicates that particle's velocity or position is referred to the observation instant τ_a while index b says that the characteristics of another particle are evaluated at the retarded moment $\tau_b^{\text{ret}}(\tau_a)$.

To construct particle's equation of motion we need an expression which explain how the 4-momentum $p_{\text{part},a}$ of charged particle depends on its individual characteristics (velocity, mass *etc*). To derive the desired expression, we analyse the angular momentum balance equations :

$$z_a \wedge \Big| \dot{p}_{\text{part},a} + \frac{2e_a^2}{3} a_a^2 u_a - F_{ba}^{\text{ret}} - e_a \dot{A}_{ba}^{\text{ret}} + \frac{1}{2} \dot{d}_{ba}$$

$$+u_a \wedge \Big(p_{\text{part},a} + \frac{2e_a^2}{3} a_a - e_a A_{ba}^{\text{ret}} + \frac{1}{2} d_{ba} \Big| = 0 . \tag{4.4}$$

Since eq. (4.3), the expression written in the first line is identically equal to zero. The solution of eq. (4.4) involves an arbitrary scalar function, say m_a :

$$p_{\text{part},a}(\tau_a) = m_a u_a - \frac{2e_a^2}{3} a_a + e_a A_{ba}^{\text{ret}} - \frac{1}{2} d_{ba} . \tag{4.5}$$

Apart from Teitelboim's expression for individual 4-momentum of a "dressed" charge [3], the 4-momentum of interacting charged particle contains also a contribution from field of another charge.

Our next task is to explain physical sense of m_a. Since the Lorentz force acted on a-th charge is orthogonal to its 4-velocity, the scalar product of u_a one the first-order time derivative (4.3) of $p_{\text{part},a}$ does not depend on F_{ba}^{ret} :

$$\left(\dot{p}_{\text{part},a} \cdot u_a\right) = \frac{2e_a^2}{3} a_a^2 + e_a \left(\dot{A}_{ba}^{\text{ret}} \cdot u_a\right) - \frac{1}{2}\left(\dot{d}_{ba} \cdot u_a\right). \tag{4.6}$$

Since $(u_a \cdot a_a) = 0$, the scalar product of particle acceleration on its 4-momentum (4.5) is as follows :

$$\left(p_{\text{part},a} \cdot a_a\right) = -\frac{2e_a^2}{3} a_a^2 + e_a \left(A_{ba}^{\text{ret}} \cdot a_a\right) - \frac{1}{2}\left(d_{ba} \cdot a_a\right). \tag{4.7}$$

Summing up (4.6) and (4.7) and taking into account eqs. (3.11) and (3.4) we obtain

$$\frac{d}{d\tau_a}\left[\left(p_{\text{part},a} \cdot u_a\right) - \frac{e_a}{2}\left(A_{ba}^{\text{ret}} \cdot u_a\right)\right] = 0 \tag{4.8}$$

after collecting like terms. Alternatively, the scalar product of 4-momentum (4.5) on particle's 4-velocity looks as follows :

$$\left(p_{\text{part},a} \cdot u_a\right) = -m_a + \frac{e_a}{2}\left(A_{ba}^{\text{ret}} \cdot u_a\right). \tag{4.9}$$

Having compared it with previous expression we are sure that scalar function m_a does not change with time. It is natural to interpret it as a finite (already renormalized) rest mass of a-th particle.

Finally, we differentiate the expression (4.5) and substitute it for the left-hand side of eq. (4.3). After cancellation of like terms we arrive at the relativistic generalization of Newton's second law

$$m_a a_a = \frac{2e_a^2}{3}\left(\dot{a}_a - a_a^2 u_a\right) + F_{ba}^{ret} \tag{4.10}$$

where loss of energy due to radiation is taken into account.

In terms of kinematical variables conservation laws looks as follows :

$$P = \sum_{a=1}^{2}\left(m_a u_a - \frac{2e_a^2}{3} a_a + \frac{2e_a^2}{3}\int_{-\infty}^{\tau_a} ds_a a_a^2 u_a - \int_{-\infty}^{\tau_a} ds_a F_{ba}^{ret}\right) \tag{4.11}$$

$$\hat{M} = \sum_{a=1}^{2} z_a \wedge \left(m_a u_a - \frac{2e_a^2}{3} a_a \right) + \frac{2e_a^2}{3}\int_{-\infty}^{\tau_a} ds_a \left[a_a^2 z_a \wedge u_a + u_a \wedge a_a \right]$$

$$- \int_{-\infty}^{\tau_a} ds_a z_a \wedge F_{ba}^{ret} \tag{4.12}$$

The work done by Lorentz forces of charges acting on one another exhausts the radiation reaction due to combination of fields. Therefore, the interference of outgoing electromagnetic waves leads to the interaction between the sources.

5. Conclusions

In the abstract of Ref. [12] the authors quoted from A. Einstein : "... the energy tensor can be regarded only as a provisional means of representating matter. In reality, matter consists of electrically charged particles ..." Teitelboim [3] explains the way it can be done. Namely, the volume integration of the energy-momentum density reveals the matter "islands" at field "sea". If one adds all the contributions from the various volume elements, they obtain terms of two quite different types. Let the net result depends only and a neighbourhood of the present event. It means that we reveal where the charged singularity is placed, renormalize its mass, and modify its 4-momentum. The net terms which depend on prior history of the source describe the radiation which escapes to infinity. They exert the radiation reaction which includes effect of particle's own field.

The volume integration the stress-energy tensor reduces field's uncountably infinite degrees of freedom. We obtain the action at a distance theory where particles interact directly with one another. Neverthless, the delay in disturbances caused by the circumstance that an action is propagated not instantaneously is taken into account. Although the theory satisfies the requirement of retarded causality it seems to resemble the Wheeler and Feynman electrodynamics [11–13].

It would be interesting to consider the specific case when the charged particles are a very close to each other. Since the electromagnetic field satisfies the superposition principle, the models either an extended object consisting of N point charges or a continuous charge distribution are based on dynamics of two-body system [14,15].

Having analysed the radiative component of mixed stress-energy tensor which is presented in Appendix we can establish an angle distribution of two-body radiation.

References

[1] P A M Dirac *Proc. R. Soc London, Ser.* **A167** 148 (1938)

[2] F Rohrlich *Classical Charged Particles* (California : Addison-Wesley, Redwood) (1990)

[3] C Teitelboim *Phys. Rev.* **D1** 1572 (1970)

[4] C A López and D Villarroel *Phys Rev* **D11** 2724 (1975)

[5] J M Aguirregabiria and L Bel *Phys. Rev.* **D29** 1099 (1984)

[6] H J Bhabha *Proc. R. Soc. London, Ser.* **A172** 384 (1939)

[7] E T Newman and T W J Unti *J Math Phys.* **4** 1467 (1963)

[8] J M Aguirregabiria and J R Etxebarria *J. Math. Phys.* **29** 1832 (1988)

[9] Yu Yaremko *J. Phys. A : Math. Gen.* **37** L531 (2004)

[10] Yu Yaremko *Int. J. Mod. Phys.* **A20** 129 (2005)

[11] J A Wheeler and R P Feynman *Rev. Mod. Phys.* **17** 157 (1945)

[12] J A Wheeler and R P Feynman *Rev Mod. Phys* **21** 425 (1949)

[13] F Hoyle and J V Narlikar *Rev. Mod. Phys.* **67** 113 (1995)

[14] A Ori and E Rosental *Phys. Rev.* **D68** 041701(R) (2003)

[15] A Ori and E Rosental *J. Math. Phys.* **44** 2347 (2004)

Appendix

Similarly to one-particle stress-energy tensor, the mixed contribution (1.6) to the Maxwell tensor can be decomposed into two parts which separately conserved away from the world lines. Bound part produces the terms which describe unavoidable deformation of rigid "clouds" carried along "bare" charges due to mutual interaction. It contributes to particles' 4-momenta of dressed charges (see eq. (4.5)). The radiative part produces the radiation which detach themselves from the charges and lead independent existence. It is the symmetric combination (3.3) of work done by the retarded and the advanced Lorentz forces of point-like sources acting on one another plus one-half of sum of the retarded and the advanced Liénard-Wiechert potentials.

Value of $\hat{T}_{\rm int,rad}$ at arbitrary point $y \in \mathcal{M}_4$ is determined by the state of particles' motion at instants τ_1 and τ_2 which label the points $z_1(\tau_1) \in \zeta_1$ and $z_2(\tau_2) \in \zeta_2$ at which the past light cone of y is punctured by the world lines of the 1-st and the 2-nd particles, respectively (see Figure 5). It have the form

$$T^{\mu\nu}_{\text{int,rad}} = \frac{e_1 e_2}{4\pi}\left(t^{\mu\nu} - \eta^{\mu\nu} t^{\alpha}_{\alpha}\right)$$

where

$$t^{\mu\nu} = \frac{(u_1 \cdot u_2)}{r_1^2\, r_2^2}\left(k_1^{(\mu} k_2^{\nu)} - u_1^{(\mu} k_2^{\nu)} - k_1^{(\mu} u_2^{\nu)} + u_1^{(\mu} u_2^{\nu)}\right)$$

$$+\frac{(u_1 \cdot u_2) a_2^k + (u_1 \cdot a_2)}{r_1^2 r_2}\left(k_1^{(\mu} k_2^{\nu)} - u_1^{(\mu} k_2^{\nu)}\right)$$

$$+\frac{(u_1 \cdot u_2) a_1^k + (a_1 \cdot u_2)}{r_1 r_2^2}\left(k_1^{(\mu} k_2^{\nu)} - k_1^{(\mu} u_2^{\nu)}\right)$$

$$+\frac{(u_1 \cdot u_2) a_1^k a_2^k + (u_1 \cdot a_2) a_1^k + (a_1 \cdot u_2) a_2^k + (a_1 \cdot a_2)}{r_1 r_2}\, k_1^{(\mu} k_2^{\nu)}$$

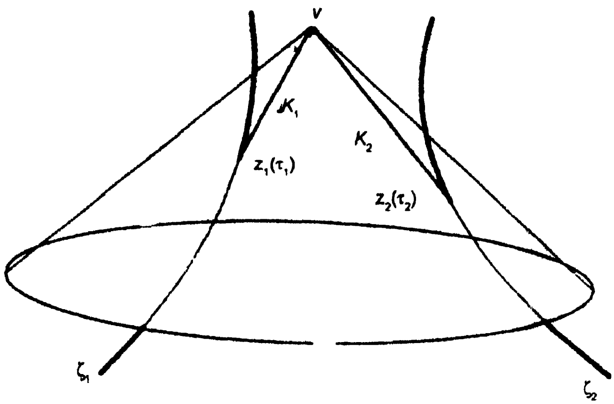

Figure 5. The past light cone with vertex at point $y \in \mathcal{M}_4$ is punctured by the world lines of the 1-st particle and the 2-nd particle at points $z_1(\tau_1)$ and $z_2(\tau_2)$, respectively. The vector K_a is a null vector pointing from $z_a(\tau_a)$ to y.

We use round brackets to denote symmetrization of the indices, e.g $k_1^{(\mu} k_2^{\nu)} = k_1^{\mu} k_2^{\nu} + k_1^{\nu} k_2^{\mu}$. Future-oriented null vector k_a is a-th null vector K_a rescaled by a factor r_a^{-1} where

$$r_a = -(K_a \cdot u_a).$$

$a_a^k = (a_a \cdot k_a)$ is the component of a-th particle acceleration in the direction of k_a. Indices 1 and 2 numerate the particles and indicate the instants at which their characteristics are evaluated.

The radiative component of the torque stress-energy tensor will be presented elsewhere.